A DISCRETE MAXIMUM PRINCIPLE SOLUTION TO AN OPTIMAL CONTROL FORMULATION OF TIMBERLAND MANAGEMENT PROBLEMS

by

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ABSTRACT

The relationship between optimal control and problems in timberland management is discussed; and the usefulness of the control theoretic approach is demonstrated. A two-part forest management model, consistent with the optimal control approach is derived: the first part consists of economic/biological objectives which are to be optimized; the second part is the physical forest evolution model. A solution algorithm based on a control vector iteration realization of the discrete maximum principle is derived, and details of its implementation are considered. The solution of a specific problem is discussed; and results are compared with those obtained previously from another model. PROSE listings and I/O are provided.

I. INTRODUCTION

The study of capital theory, like other branches of economics, has in the past been constrained to the consideration of stationary equilibrium. Capital theory, however, is concerned with the growth of capital over time. Until recently the study of this process was confined to a "snapshot" approach as stationary equilibrium models were applied sequentially in an effort to gain some insight into the dynamics of growth. Recently developments in optimal control theory have allowed the consideration of the dynamics of capital growth; and it can be shown that optimal control theory is formally identical with capital theory, and that its main insights can be attained by strictly economic reasoning. An excellent economic interpretation of optimal control theory is presented in [1].

The basic problem facing any manager responsible for the management of resources and assets is how to best manage these resources to attain the objectives of the organization or his constituency. In general, this objective is the maximization of wealth or benefit over a given planning horizon. It will be shown that this problem is correctly approached using optimal control theory in that it provides a truly optimum solution and specifies the policies that must be implemented within the planning horizon to attain that solution.

The manager of public or private forest lands is faced with the above problem.

The purpose of this paper is to show the usefulness of optimal control theory in assisting forest managers to optimize their management decisions, and to present and discuss an optimal control model for determining these decisions. Besides allowing the consideration of nonlinear demand and cost functions, and the application of constraints to discrete time periods or segments of the assets managed, optimal control theory provides the forest manager with a truly optimal solution consistent with his objective and constraints. Other nonlinear programming techniques generally do not obtain an optimum time path for management action. In general the solution technique used in nonlinear approaches involves the application of a static optimization technique sequentially from period to period within the horizon, resulting in a suboptimal solution. The impact of actions taken in subsequent periods is not adequately considered by this solution technique. In the optimal control approach to forest management, the entire planning horizon is considered as an entity; and the impact of decisions on subsequent periods is thus taken into account.

We begin with a brief survey of the work previously done in forest modeling, The treatment is not intended to be exhaustive, nor is it limited strictly to optimal control models. Following this discussion we present a general discrete optimal control initial value problem and show its relationship to forestry problems, thus demonstrating the appropriateness of the optimal control approach for the solution of timberland management problems. After this we derive the two-part model which is the main concern of this work: the Timberland Investment Management Model (TIMM). TIMM consists of a physical forest evolution model, which keeps track of the planting, growing, and harvesting of the trees, and an economic model describing the financial objectives of the forest manager. The physical forest model is essentially invariant to changes in the economic model, and this provides a high degree of flexibility in the construction of the economic model. Also, it is possible to include biological goals in the economic objective, in the form of penalty functions, or additional transfer functions.

After the model has been established, we discuss the method used to solve it. The underlying solution structure is embodied in the discrete maximum principle, which is closely related to the Maximum Principle of Pontryagin [2]. (The exact nature of this relationship is discussed in Appendix A.) Except in the simplest cases the discrete maximum principle, per se, does not provide a solution to nonlinear problems; some iterative technique must be used in addition. There are a number of possibilities; the method employed here is known as control vector iteration. The particular implementation of discrete maximum principle/control vector iteration used to solve TIMM is known as EPOC, an acronym for Economic Planning via Optimal Control. It is a general solution algorithm for discrete optimal control initial value problems, coded in PROSE for the CDC 6600 computer. Use of the PROSE language leads to automatic generation of the partial derivatives (a feature not available in other languages) required by the solution technique, thus eliminating the need to derive and code derivative formulas. The PROSE listing of EPOC/TIMM is included as Appendix B.

Having developed a solution technique, we present results obtained using a modified version of the model due to Walker [3]. The results obtained from EPOC/TIMM show both similarities and dissimilarities to those of [3]. The results of the present model are interpreted economically, and compared with those of Walker.

Survey of Previous Work

There are already a large number of forest management models, both linear and nonlinear; and it is beyond the intended scope of this paper to provide more than a scant review of any of them. For the most part, the usefulness of any one of these models is generally limited to a narrow range of forestry problems; and in many cases the solutions obtained even to problems within this range are not altogether satisfactory.

As would be expected, there are many linear models. Among these are the models discussed in a compendium of such models, due to Johnson and Scheurman [4], the model of Nautiyal and Pearse [5], and the models Timber RAM and MAX-MILLION. A blanket criticism which can be made of such models is simply that they are linear; and the natural formulation of many forest management problems is inherently nonlinear. One way to circumvent this difficulty is to piecewise linearize the nonlinear problem, and solve each piece with an LP model. We discuss some consequences of this in Section V.

Nonlinear models also abound; and most, but certainly not all, of these are formulated as optimal control problems. Many of these have used dynamic programming (cf. Bellman [6]) as the solution technique, which as it is well-known, is handicapped by the "curse of dimensionality." Three examples of such models are those of Amidon and Akins [7], Hool [8], and Schreuder [9]. In the first of these optimal levels of growing stock for an even age class forest were determined for age classes of length 5 years, over a 65-year rotation period. However, no details of either the economic model or the computational results are reported. The model [8] is nondeterministic. It employs dynamic programming to determine control activities in each planning period, for a range of possible conditions of the forest. At each period probabilities for attaining a given condition are prescribed; and optimization takes place on the most probable conditions. The model was solved for a small example problem. In [9] dynamic programming was used to determine optimal thinning and rotation schedules for an even age class forest. This was done analytically; no computational results were reported.

Näslund [10] used the maximum principle to solve the same problem as solved by Schreuder; but without the even-age forest assumption. Sethi [11] uses the discrete maximum principle to determine a fertilization program with which to optimize the present value of the forest. No computational results are reported in either of the above cases.

In [3] the net present value of a forest is optimized over a fixed planning horizon while accounting for decreasing demand with increasing volume, and distributing the harvest between two markets. While the above mentioned nonlinear models are optimal control problems, the model given in [3] is not. It does not contain separately identifiable controls, states, and objectives as a typical control formulation would; and the solution technique is rather ad hoc. Nevertheless, reasonable results were obtained, and the basic model has since been extended to the model known as ECHO (Economic Harvest Optimization).

Optimal Control and Its Application to Forestry

A typical optimal control problem consists of the following items:

(a) state variables (including initial and/or boundary states)

(b) control variables

(c) objective function

(d) transfer functions

The goal of an optimal control problem is to move some given system from one point (in space, time, or both - or maybe neither) to another in the "best" possible manner, where "best" is measured in terms of the value of the objective function. Moving the system from one point to another involves inducing a change in various of the attributes which characterize the system. Collectively, these attributes are usually termed the "state" of the system; and they simply provide some, hopefully, meaningful quantitative description of the system being studied. An individual attribute is called a state variable. Changes in the system state are caused by applying one or more of the controls, or control variables. (These are often called "decision" variables or "policies.") These control variables appear explicitly in the transfer functions (known also as state equations) which quite literally transfer the system from one state to a succeeding one. The objective function depends explicitly on the state of the system and thus implicitly on the controls; but it may also depend explicitly on the control variables, as well.

Since we will later be dealing exclusively with a discrete version of the optimal control approach, the optimal control equations given below will be in discrete form.

A typical optimal control initial value problem has the following form:

maximize

$$P = \sum_{n=1}^{N} P_n(X_{n-1}, U_n) + G(X_N) \tag{1}$$

subject to

$$X_n = F_n(X_{n-1}, U_n) \quad \text{with } X_o \text{ given,} \tag{2}$$

and

$$U_{n,min} \leq U_n \leq U_{n,max} \tag{3}$$

X_n, U_n, and F_n are vectors; and the inequalities in (3) clearly must apply to corresponding components. It is hoped this abuse of notation will not be overly confusing, as it is quite convenient. Equation (1) is the objective function, written as a sum of objectives for each period, plus a term depending only on the final system state. Equation (2) is a vector equation representing the totality of transfer functions which cause the system to progress from state n-1 to state n due to the application of the controls, U_n. Finally, the inequalities (3) represent a component of the problem which precludes the use of the calculus of variations, and forces the use of some form of the maximum principle or dynamic programming. That is, the control variables are constrained; and it was a need to solve such problems which led to the development of the maximum principle and dynamic programming by Pontryagin and Bellman, respectively. (A good comparative treatment of these two approaches can be found in Intrilligator [12].)

Forest management problems, and in fact, many resource management problems in general, fit into the above framework quite naturally, as we will now show. Obviously, the forest problem has some goal or objective which must be met. This objective is nearly always of an economic nature, for example maximization of net present value over a planning horizon of N periods; but additional, usually biological, goals may also be involved. A typical example is the conversion to even flow harvest (cf. Nautiyal and Pearse [5]). Most objectives of these types can readily be expressed in the form of (1).

The forest, being a physical system, can be represented in terms of some set of physical attributes. Quite a variety of quantities have been used to characterize the forest in the various earlier models. In the model presented here, we have chosen to express the state of the forest in terms of the age class/type site area distribution. By this, we mean that each state variable (and its corresponding transfer function) corresponds to the area of land of a particular type (flat, hilly, etc.) forested by a particular species of trees of a specific age.

Finally, the evolution of a managed forest is controlled by various activities such as thinning, harvesting, and planting. Because we have described the state in terms of very specific areas of tree types and ages we can define the controls as the intensity of the various management activities applied to each of these areas. This completes the specification of the optimal control formulation.

As can be seen, each major portion of the forest management problem fits naturally into one of the parts of a typical optimal control problem. The preceding discussion is summarized in Table 1.

TABLE 1

CORRESPONDENCE OF FOREST PROBLEM WITH OPTIMAL CONTROL FORMULATION

Forest Problem Component	Control Formulation Component	
Economic/biological goals	Objective function, P:	Equation (1)
Forest inventory at period n (age class/type site area distribution)	State variables, X_n:	Equation (2)
Intensity of management activities during period n (Regeneration, fertilization, thinning, harvesting, etc.)	Control variables, U_n:	Inequality (3)

II. THE MODEL

As mentioned above, EPOC/TIMM consists of two parts: 1) economic/biological objectives, 2) forest state evolution. We will begin by discussing, rather generally, the formulation of the economic portion of the model. (A specific model will be discussed in more detail in Section IV.) We then develop the equations of the physical forest.

Economic/Biological Objectives

In considering the appropriate objective function to utilize in EPOC/TIMM, the position or charter of the forest manager must be considered. The objective of the manager of privately owned forest land may be to maximize the return on his investment, where discounted net cash flow is the appropriate measure of efficiency. The manager of public lands, although not exempt from economics, may have to consider a constituency beyond stockholders, customers, and employees when optimizing the utilization of the resources under his responsibility. In this case the appropriate objective function would be modified to reflect these additional constraints. An excellent review of the economic aspects of various objectives used in forest management is given in [13].

The objective function utilized in the present case is given by

$$P = \sum_{n=1}^{N} \frac{R_n - C_n}{(1 + r)^n} , \qquad (1)$$

which is simply the equation to determine the net present value, P, of an investment over discrete time periods. Here R_n and C_n are revenue and costs, respectively, for period n; and r is the discount rate.

In the present work we will ignore the problems encountered with the above when capital rationing and rising marginal cost of capital occur. An adequate discussion of these effects is presented in [14]. The objective function utilized, therefore, is oriented toward the private owner of forest land. However, two points should be noted. First, the appropriate use of penalty functions in equation (1) and/or constraints on the values of controls (e.g., harvest, regeneration levels, etc.) over the planning horizon can be used to reflect the management problem faced by the manager of public forest lands. Secondly, the utilization of the model without these penalty functions or control constraints can be used to measure the cost of any benefits derived from the noneconomic use of forest lands (e.g., recreational uses, flood control, etc.). Therefore, cost/benefit tradeoffs can be accomplished, using the model with the present objective function. In general any objection function can be substituted for equation (1), constrained only by the condition that the variables in the objective be related to the state and/or control variables of the physical forest model.

In evaluating (1) we consider the terms in the summation separately. In forest management problems, the costs arise from several main sources. These are: 1) regeneration, 2) harvest and thinning, 3) fixed costs. Others could certainly be considered, but for the present we will assume that any other costs may be included in one of the three listed above. Thus, we have

$$C_n = C_{R,n} + C_{H,n} + C_{F,n}. \tag{2}$$

Each of the terms in (2) may be computed using whatever degree of detail is felt necessary or desirable within the framework of the particular problem being studied. (For example, harvesting and thinning costs may be distinguished.) In general, we expect that the regeneration costs should depend on the acreage regenerated and possibly also on the type of land and species of trees, i.e., the type site, involved. Harvest costs depend upon the volume of timber harvested, but more specifically on the volume per acre. A typical harvest cost function is given in [3], and in fact, will be used in the model discussed in Section IV. The shape of the harvest cost function has a significant impact on the optimal solution. The function, shown schematically in Figure 1 (taken from [3]), relates the cost per unit volume of timber harvested to the volume per acre of the stand (stand yield). From the shape of the curve it can be inferred that the unit cost of harvest decreases with increasing timber density, as the "fixed" costs associated specifically with harvesting are amortized over larger volumes. The fixed costs term provides a convenient term in the cost function for including such things as investment in additional acreage and equipment, general overhead, etc. It should be noted that $C_{F,n}$ as well as the other two terms, may depend on the specific planning period under consideration; and even the mathematical form of these functions is permitted to change from period to period in the present model. That is, shifts in plant cost functions due to technological advances and/or capacity changes can be included.

The cost functions used in the present EPOC/TIMM model, as noted above, have been taken from [3]. These functions do not include a cost penalty for significantly decreasing the harvest volume. In practical applications, the decision maker is generally faced with the problem of utilizing a plant and/or equipment which is designed for an optimum capacity, where the unit cost of processing increases significantly with increases or decreases in utilization. This effect is shown graphically in Figure 1. The results presented in this paper would be noticeably altered if a similar relationship had been included. In order to provide a comparison to prior work, however, the cost functions in [3] were not revised.

The revenue in period n will depend generally on the volume of timber marketed in that period. Moreover, the price paid per unit volume may be assumed to be influenced by this volume; and, of course, it could be expected that prices (and even the level of demand) might vary from period to period. Thus, an implicit expression for the revenue in period n is

$$R_n = R_n(V_{H,n}, p(V_{H,n}, n)). \tag{3}$$

In this equation, $V_{H,n}$ is the total volume of timber harvested in period n, and p is the price paid per unit volume. This can be generalized to include separate prices for thinning and harvesting.

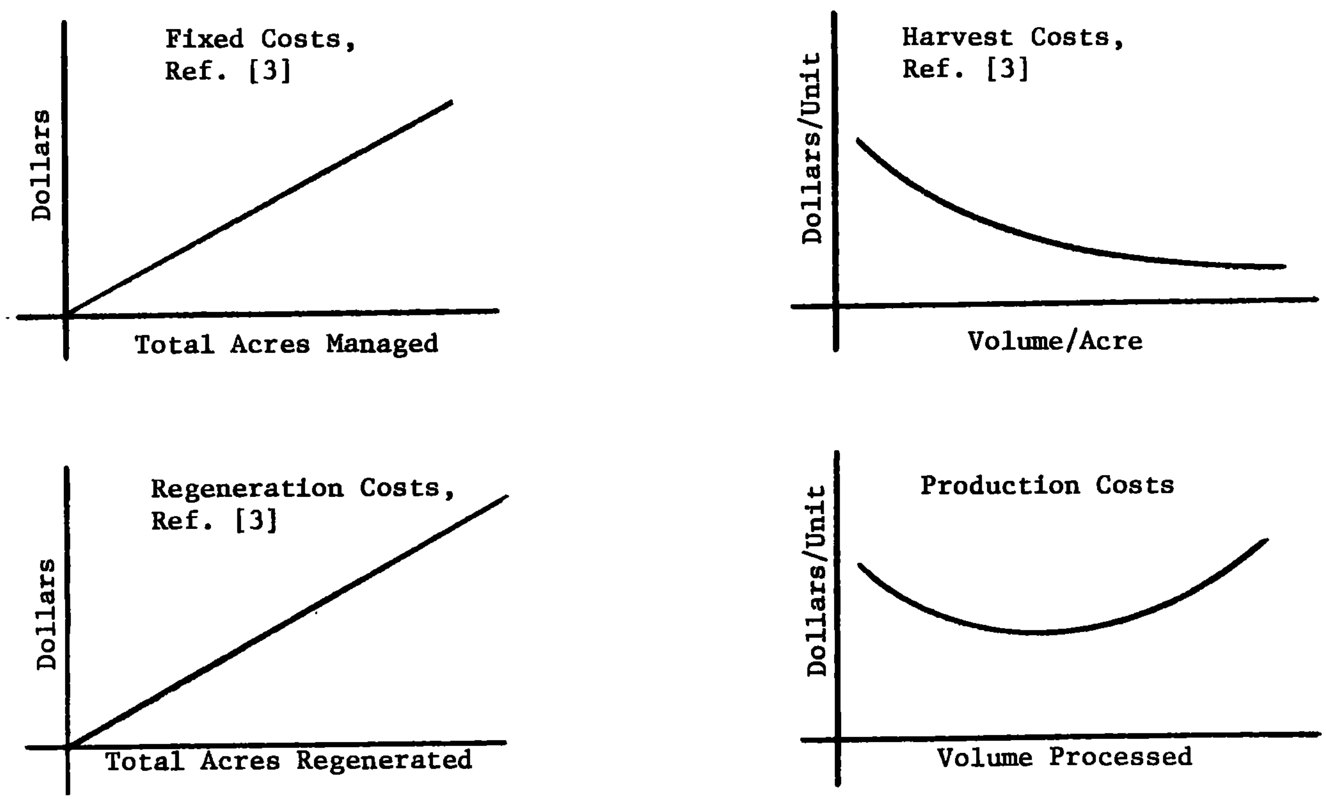

Figure 1. EPOC/TIMM Forest Management Cost Functions

In constructing a specific model some thought should be given to the appropriate discount rate, r, an individual decision maker should utilize when ranking potential investment in private or public forest lands. Certainly the owner of private forest lands has greater direction and data on which to base this decision since the private firm competes for capital in both money and equity markets. An adequate discussion of the methods a private firm can utilize in measuring its past and present performance in these markets to determine a measure of its cost of capital, and perhaps an appropriate discount rate, is presented in [14].

The manager of public forest lands is faced with a more poorly defined problem in determining the appropriate rate to utilize. There is certainly an opportunity cost associated with the investments made in public forest lands. Discount rates are now utilized by various government agencies in completing cost/benefit analyses of alternative investments. The Office of Management and the Budget is perhaps the best source of information to utilize in deriving an appropriate discount rate.

Certain assumptions are inherent in any approach to economic modeling. The following are implicit in the foregoing development, and should be noted.

- The existence of capital rationing or rising marginal cost of capital are assumed not to apply. Adjustments required for these cases are considered in [14].
- Revenue received from the sale of timber products is the measure of benefits derived from the investment in forest land. However, the opportunity cost (net present cost) of noneconomic objectives can be quantified by the model.
- The net cash flow derived from the sale of timber products in a given period is reinvested at the discount rate utilized in the net present value calculation.
- Timber harvested or thinned in a period is assumed to be processed and sold in the same period (i.e., inventory levels are not considered).

Before proceeding to the derivation of the equations representing the physical forest, we briefly discuss two more topics: multiple type sites and resource allocation.

The problem of multiple type sites (defined for this model as any segmentation of the acreage being managed other than age classes) is easily handled in EPOC/TIMM. Differentiation of various type sites in the economic model can be accomplished if different cost functions and/or markets apply for different type sites over the planning horizon or for a given period within the horizon. An important point to remember is that the solution technique optimizes the solution of all type sites over the horizon subject to the given objective function and constraints.

The utilization of various type sites in EPOC/TIMM allows the manager of public forest lands to differentiate land segments by use (i.e., recreational, flood control, etc.) and to more easily define constraints on the control levels for particular type sites over the planning horizon. In this way EPOC/TIMM provides the possibility of leaving land parcels intact throughout a planning horizon as do the Type I LP models discussed in [4].

Another important economic consideration in the management of forest lands is the efficient allocation of resources between type sites and/or the allocation of forest products amongst competing markets. This problem has been addressed using linear programming techniques in [15]. The important economic difference between the solution technique utilized in [15] and that of EPOC/TIMM is that in [15] the harvest levels for various production regions are given; and the allocation of forest products to markets, and the utilization of resources to provide these products, proceeds from these known harvest levels. In EPOC/TIMM the optimum harvest level is directly influenced by the demand and cost functions of the various markets, resulting in a true optimal solution.

The Physical Forest Model

The physical forest model utilized in TIMM, although it accomplishes the same goals as do various other forest models, is constructed much differently than are most of its predecessors. It is composed simply of transfer functions for the optimal control algorithm, and nothing more. In this respect it is extremely compact and simple, as a glance at MODEL .TRANSFER in the PROSE code, Appendix B, will indicate. At the same time, the model is quite general, and powerful; it faithfully reproduces the various activities associated with forest management, and their effects on the forest. It permits the determination of not only when various management activities should be applied, but also to what extent, so as to achieve desired management goals. Moreover, it allows one to constrain the intensities of any one, or all, of the management activities in any period, or in every period.

Above all, the generality of the physical forest model permits the study of a wide range of economic/biological objectives without change to the forest model itself. This is exactly what one should expect if a model is to provide a faithful representation of a forest, for the forest does not know of man's economic objectives. It simply grows from year to year, affected only by the activities imposed by those who manage it, and by the forces of nature. (The latter is not considered in the present model.)

The transfer functions were introduced formally in equation (2) (Section I). We will in this section give their explicit representations. As we have remarked earlier, the transfer functions simply move the state of the forest to the next succeeding state by applying the controls (management activities) to the present state. Hence, the left hand side of (2) (Section I) is the state of the forest at the end of period n during which controls U_n have been applied. For a forest consisting of I age classes and

J type sites, the state vector X_n for the nth period may be expressed in terms of its components as

$$X_n = (x_{1,1,n}, x_{2,1,n}, \ldots, x_{I_1,1,n}; x_{1,2,n}, x_{2,2,n}, \ldots, x_{I_2,2,n}, \ldots \\ \ldots; x_{1,J,n}, \ldots, x_{I_J,J,n}) \qquad n = 0,1,\ldots,N \tag{1}$$

The subscripts on the age class indices serve as an indication that different type sites may be composed of different numbers of age classes. Each component of X_n has associated with it a transfer function $f_{ij,n}$ so that

$$x_{ij,n} = f_{ij,n}(X_{n-1}, U_n) \tag{2}$$

From the form of the arguments of the function $f_{ij,n}$ the ijth state at period n may be determined by other than the ijth state at period n-1. The regeneration state $x_{1,j,n}$, is an example of this (see below).

The control vector U_n, first given in the inequalities (3) of Section I, also may be written in terms of its components:

$$U_n = (u_{1,1,n}, \ldots, u_{I_1,1,n}; \ldots; u_{1,J,n}, \ldots, u_{I_J,J,n}) \quad n = 1,\ldots,N \tag{3}$$

It is important to notice the one-to-one correspondence (ignoring the initial state components) between the components X_n and those of U_n. The implication of this is that for any particular age class/type site parcel at period n, one and only one management activity is permitted. This does not preclude the possibility of no activity at all, since this case may be expressed mathematically as a zero intensity of any of the possible activities. The admission of only one management activity per age class/type site, per period, is for convenience only. It has nothing to do with model structural considerations; but on the other hand, it is felt that this should supply adequate management control. At any rate, the restriction can be lifted if this is felt necessary.

Before going on to the explicit representations of the transfer functions it should be remarked that the other members of the inequality (3) of Section I have representations analogous to (3) in the present section. The significance of this is that each component of the control vector U_n is bounded above and below; and this provides the user of the model with a high degree of control over the specification of intensity of activity to be applied to each segment of the forest in each planning period.

The transfer functions of the current forest model reflect application of three basic management activities:

(a) Regeneration

(b) Thinning

(c) Harvest

However, unlike the version of the model reported earlier in [16], thinning is merely a special case of harvesting (the opposite of the approach taken in [10]). This leads to a great simplification to the transfer function associated with regeneration; and it is felt that no great sacrifice of model flexibility has been incurred. The basic assumptions used in constructing the current version of the model are essentially the same as those given in [16], except that it is now possible to drop the last assumption used in that development. Namely, we have:

(a) The time spanned by any age class coincides with the length of one economic planning period.

(b) Every age class may be thinned or harvested, with the exception of the first, which is vacant land.

In contrast to the earlier treatment, land may or may not be regenerated immediately after harvesting. This decision can be left to the solution algorithm, or it may be made by the user of the model.

Each component $x_{ij,n}$ of the vector X_n represents the area (or volume, or number of trees)* in age class i (d.b.h. class i) on type site j in period n. In achieving this state, particular components of U_n are applied to certain components of the state vector X_{n-1} representing the forest state in period n-1.

It is clear from (1) that each type site has its own set of transfer functions, so the "j" subscripts will be deleted, for notational clarity, from the following development with the understanding that the results obtained must be repeated for each of the J type sites.

In arriving at the regeneration transfer function the following items must be considered: amount of land available for regeneration, fraction of total to be regenerated, and amount of land returned to the regeneration pool due to harvest and thinning.

The amount of land available at the end of period n for regeneration beginning in period n+1 consists of two components. First, at the beginning of period n, there are $x_{1,n-1}$ units of vacant land left from the end of period n-1. During

*In the present discussion we shall always consider age class/type site area distributions. However, it is quite possible to formulate a problem in terms of volumes or numbers of trees, and to use d.b.h. classes rather than age classes.

period n some fraction (determined either a priori due to some constraint and supplied as input data, or by the solution process) of this vacant land will be regenerated. Indeed, this fraction is precisely the regeneration control variable for period n, $u_{1,n}$. Because $u_{1,n}$ is a fraction it satisfies the inequalities

$$0 \le u_{1,n} \le 1; \tag{4}$$

but more generally,

$$u_{1,n,min} \le u_{1,n} \le u_{1,n,max} \tag{5}$$

where $u_{1,n,min}$ and $u_{1,n,max}$ each satisfying inequalities such as (4) separately; and in addition

$$u_{1,n,min} \le u_{1,n,max} \ . \tag{6}$$

After the control $u_{1,n}$ is applied to $x_{1,n-1}$, the area remaining for regeneration in period n+1 is $x_{1,n-1}(1-u_{1,n})$. But there is a second component of area which will contribute to the size of the regeneration land pool. This comes from the harvesting and thinning of land during the nth period. Since thinning is now a special case of harvesting, the area vacated by these processes may be expressed simply as

$$\sum_{k=2}^{I} x_{k,n-1}u_{k,n},$$

where inequalities such as (4), (5), and (6) must be satisfied by each $u_{k,n}$; and I is the maximum tree age (more accurately, maximum age class index) for the particular type site under consideration. Thus the total area available for regeneration at the beginning of period n+1 (end of period n) is

$$x_{1,n} = x_{1,n-1}(1 - u_{1,n}) + \sum_{k=2}^{I} x_{k,n-1}u_{k,n}. \tag{7}$$

It is clear that the amount of land regenerated in period n is just $x_{1,n-1}u_{1,n}$. Applying assumption (a) we see that this must move into period n+1 as age class 2 area. Hence, the area in age class 2 at the end of period n is

$$x_{2,n} = x_{1,n-1}u_{1,n} \ . \tag{8}$$

As the model is presently formulated there is just one other form of transfer function. This one form can account for thinning and harvesting in all age classes. While assumption (b) was used in obtaining (7), we again appeal specifically to assumption (a) in constructing the remaining transfer function, as was done for equation (8). At the end of period n-1 (beginning of period n) there will be $x_{i-1,n-1}$ units in age class i-1. During the nth period a fraction $u_{i-1,n}$ of this amount will be harvested. The remainder will grow on into period n+1. But it can no longer be in age class i-1; because of assumption (a) it will have aged by one age class. Thus, at the end of period n, the units of area in age class i must be

$$x_{i,n} = x_{i-1,n-1}(1 - u_{i-1,n}) \qquad i = 3,4,\ldots,I. \tag{9}$$

We note that if no harvesting is done, i.e., $u_{i-1,n} = 0$, all of the area just moves into the next age class in the succeeding planning period.

Besides their simplicity, the transfer functions formulated as done here automatically conserve the area of land on which the forest enterprise takes place. Thus, no acreage constraints such as those found in the various LP models need be imposed. On the other hand, as will be seen in the next section, the equations (7), (8), and (9) are, at least formally, treated as constraints in the solution algorithm. However, the implication of this is rather different here than would be the case in static nonlinear programming.

III. SOLUTION TECHNIQUE

The Discrete Maximum Principle

The solution method applied to the above model is known as the discrete maximum principle which, as noted earlier, is related to the Maximum Principle of Pontryagin. Both discrete and continuous versions of the maximum principle have been used previously in solving forest management models. A large number of discrete problems from various fields are solved analytically in [17], one of the basic references for the discrete maximum principle. However, the treatment given here follows more closely that of [18].

Equations (1), (2) and inequalities (3) of Section I form the basic problem structure for development of the discrete maximum principle; we shall repeat these here for ease of reference.

$$P = \sum_{n=1}^{N} P_n(X_{n-1}, U_n) + G(X_N) \tag{1}$$

$$X_n = F_n(X_{n-1}, U_n), \text{ with } X_o \text{ given;} \tag{2}$$

$$U_{min,n} \leq U_n \leq U_{max,n} \qquad n=1,2,\ldots,N \tag{3}$$

The transfer functions, equation (2), can be treated formally as equality constraints by writing

$$X_n - F_n(X_{n-1}, U_n) = 0 \,. \tag{4}$$

This vector equation is then adjoined to the objective function, equation (1), by employing a vector of Lagrange multipliers λ_n often called co-variables or co-state variables.* Thus, corresponding to (1), we obtain the Lagrangian

$$L(X,U;\lambda) = \sum_{n=1}^{N} \{P_n(X_{n-1}, U_n) - \lambda_n[X_n - F_n(X_{n-1}, U_n)]\} + G(X_N) \,. \tag{5}$$

In this form the problem is no different, conceptually, than problems in static optimization; and we are thus led to seek necessary conditions for an optimum in the same manner in which we would for that case. Namely, we set to zero the partial derivatives of L with respect to X,U and λ.

*These names arise from the quite deep, but beautiful, mathematical theory underlying this approach (cf., Abraham, Foundations of Mechanics). But, we shall here refrain from the use of such terminology in favor of the more familiar, Lagrange multiplier.

This is formally equivalent to applying the Kuhn-Tucker theorem to a problem having only equality constraints. Since X,U, and λ are all vectors, we differentiate L with respect to the components.*

$$\frac{\partial L}{\partial X_{n-1}} = \frac{\partial P_n}{\partial X_{n-1}} + \lambda_n \frac{\partial F_n}{\partial X_{n-1}} - \lambda_{n-1} = 0 \qquad n = 2,3,\ldots,N \tag{6a}$$

$$\frac{\partial L}{\partial X_N} = - \lambda_N + \frac{\partial G}{\partial X_N} = 0 \tag{6b}$$

$$\frac{\partial L}{\partial U_n} = \frac{\partial P_n}{\partial U_n} + \lambda_n \frac{\partial F_n}{\partial U_n} = 0 \qquad n = 1,2,\ldots,N \tag{7}$$

$$\frac{\partial L}{\partial \lambda_n} = X_n - F_n = 0 \tag{8}$$

We immediately notice that (8) is just equation (4), implying that the constraints are satisfied. Furthermore, if we rewrite (6a) and (7) as

$$\frac{\partial}{\partial X_{n-1}} [P_n + \lambda_n F_n] - \lambda_{n-1} = 0$$

$$\frac{\partial}{\partial U_n} [P_n + \lambda_n F_n] = 0$$

we are led, naturally, to an important construct known as the Hamiltonian; that is,

$$H_n(X_{n-1},U_n;\lambda_n) = P_n(X_{n-1},U_n) + \lambda_n F_n(X_{n-1},U_n) \qquad n = 1,2,\ldots,N \tag{9}$$

Thus the conditions (6a) and (7) become, respectively,

$$\lambda_{n-1} = \frac{\partial H_n}{\partial X_{n-1}} , \tag{10}$$

$$\frac{\partial H_n}{\partial U_n} = 0 . \tag{11}$$

Equation (10) is often called the adjoint equation; (6b) provides a starting value for a backward recursive evaluation of (10):

$$\lambda_N = \frac{\partial G}{\partial X_N} . \tag{12}$$

*It should be noticed that again we have been lax with notation. By the Kuhn-Tucker theorem (6a), (6b), (7), and (8) must hold <u>at the solution</u>.

Although it is not necessary for the development here, it is of some interest to notice that since (2) holds, we can write

$$X_n = \frac{\partial H_n}{\partial \lambda_n}, \tag{13}$$

using (9). This should be compared with (10). It is the symmetry (actually skew-symmetry in the continuous case) of these two expressions which leads to the term "adjoint" for equation (10).

So far we have done nothing with the inequalities (3). We claim that the necessary conditions already obtained above apply also to any combination of inequality constraints on the states and/or controls. To see this, we consider a general inequality constraint

$$g_n(X_{n-1}, U_n) \geq 0 . \tag{14}$$

If this inequality holds, then there exists a positive slack variable Z_n^2 such that*

$$g_n(X_{n-1}, U_n) - Z_n^2 = 0. \tag{15}$$

Equation (15) is in the same form as (4) and can thus be included in the Lagrangian (5) by utilizing an additional Lagrange multiplier. Moreover, we see that Z_n can be considered as merely another state variable with $\sqrt{g_n}$ the corresponding transfer function. Hence, equations (10) and (13) can be used to determine values of λ_n and Z_n respectively, just as for X_n. Clearly, the inequalities (3) are a special case of (14), with no dependence on X_{n-1}, since we can write

$$U_n - U_{min,n} \geq 0, \quad U_{max,n} - U_n \geq 0 .$$

The preceding development serves as a proof of the following:

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*Because we assume that the problems with which we deal are nonlinear, no harm is done to the problem structure by utilizing a nonlinear, rather than linear slack variable. This technique has proved valuable in both theoretical and computational developments. The main advantage gained from use of the nonlinear term is that it may be chosen so as to be nonnegative automatically; and the additional constraints $Z \geq 0$ need not be included in the problem structure, in contrast to the linear case.

Theorem (Discrete Maximum Principle)

In order that the sequence of vectors $\{U_n\}$ be the optimal controls corresponding to the control problem given in (1), (2) and (3), the following conditions must obtain:

either

i) $$\frac{\partial H_n}{\partial U_n} = 0, \qquad n = 1,2,\ldots,N$$

or

ii) U_n is such that H_n is a maximum

iii) There exist Lagrange multipliers satisfying

$$\lambda_N = \frac{\partial G}{\partial X_N} \quad \text{and} \quad \lambda_n = \frac{\partial H_{n+1}}{\partial X_n} \qquad n = 1,\ldots,N-1.$$

(iv) $$X_n = \frac{\partial H_n}{\partial \lambda_n} \qquad n = 1,\ldots,N.$$

Some remarks are in order concerning this theorem. Once again we stress that these are necessary conditions. Hence, even when they are satisfied, an optimal solution may not have been obtained; but even more significant is the fact that if they are not satisfied, the corresponding solution generally cannot be optimal. In particular, condition iii) is often neglected in the decomposition of large problems, and in such cases it cannot be assumed that the solution is optimal. The Lagrange multipliers can be thought of as the "glue" which holds the decomposition together; and if they do not exist, the decomposed problem cannot yield the correct result for the complete "undecomposed" problem.

Condition i) above, is analogous to the usual first derivative condition from ordinary calculus. The notation of i) is somewhat strained since U_n is a vector; and i) actually implies $\sum_{j=1}^{J} jI_j$ conditions, in light of the development in Section II. In the event that the maximum of H_n occurs at control boundaries, ii) must be applied. Condition iv) as stated above implies nothing more than the satisfaction of the equality constraints of equation (4); hence it also implies $\sum_{j=1}^{J} jI_j$ conditions. Of course the incorporation of inequalities such as (14) leads to more conditions; and while this provides a convenient theoretical tool, in practice other methods may also be used, particularly when only inequalities of the form (3) are involved.

Control Vector Iteration

It is essential to realize that in most cases maximum principles provide only necessary conditions for the existence of a solution. Hence, two more elements of a solution algorithm are needed. These are sufficient conditions for the existence of an optimum, and some concrete solution procedure. (These may often be combined, as is done in the algorithm to be discussed below.) If the problem is convex, then the necessary conditions are, in fact, sufficient; and if the problem structure is fairly simple it might actually be possible to solve directly the system of equations implied by the conditions of the preceding theorem. However, this is not typically the case; and some iterative procedure is usually needed.

For initial value problems such as that described by equations (1), (2), and (3) efforts to satisfy the above conditions lead quite naturally to a solution technique known as control vector iteration, or iteration in policy space as it is sometimes termed in an operations research context. The basic steps in such an approach are: 1) guess U_n for all $n=1,\ldots,N$; 2) use these in equation (2) to determine all X_n's; (3) evaluate the P_n's; 4) calculate λ_N from $\partial G/\partial X_N$, and then calculate H_N; 5) successively evaluate λ_n, $n=N-1,\ldots,1$, using $\lambda_n = \partial H_{n+1}/\partial X_n$.

What one really is trying to find is an improved value of U_n. As each λ_n, and hence H_n, is calculated it is possible to obtain $\partial H_n/\partial U_n$. Then using the usual gradient stepping technique from static optimization,

$$U_n^{(m)} = U_n^{(m-1)} + \varepsilon \frac{\partial H_n}{\partial U_n}^{(m-1)} , \qquad (16)$$

where m denotes iteration number.

Clearly if $\dfrac{\partial H_n}{\partial U_n}^{(m-1)} = 0$, i.e., if 1) is satisfied, then $U_n^{(m)} = U_n^{(m-1)}$; and convergence will have been achieved. In (16) ε is a step length, which can be calculated by any one of a variety of schemes. We present a particular one below.

With the new values of U_n obtained from (16) the procedure is restarted unless $\partial H_n/\partial U_n = 0$ for all $n=1,\ldots,N$, in which case the optimal controls have been found.

We can summarize the preceding descriptive formulation in a formal algorithm which forms the basis for the code presented in Appendix B.

Algorithm (Control Vector Iteration)

1) Guess values of the control vector U_n for all periods $n=1,2,\ldots,N$.

2) Use the above guesses and the initial condition X_o to evaluate

$$X_n = F_n(X_{n-1},U_n) \text{ and } P_n(X_{n-1},U_n) \qquad n = 1,\ldots,N.$$

3) Using X_N obtained above, calculate

$$\lambda_N = \frac{\partial G}{\partial X_N} \; .$$

4) For $n=N,\ldots,2$ form the nth Hamiltonian

$$H_n = P_n(X_{n-1},U_n) + \lambda_n F_n(X_{n-1},U_n),$$

and calculate

$$\frac{\partial H_n}{\partial U_n} \text{ and } \lambda_{n-1} = \frac{\partial H_n}{\partial X_{n-1}}$$

5) Test whether $\partial H_n/\partial U_n = 0 \qquad n=1,\ldots,N$

If yes, solution is complete

If no, continue to 6).

6) For each n such that $\partial H_n/\partial U_n \neq 0$ calculate

$$\varepsilon_n = \min_{\substack{i\varepsilon I_j \\ j\varepsilon J}} \left| \frac{\partial H_n}{\partial u_{ij,n}} \right| \min_{\substack{i\varepsilon I_j \\ j\varepsilon J}} \left[\left|u_{ij,n}\right| + \Delta u_{min} \right] \left\| \frac{\partial H_n}{\partial U_n} \right\|_2^{-1} \tag{17}$$

where min $|\cdot|$ is taken over those i,j such

that $\left|\frac{\partial H_n}{\partial u_{ij,n}}\right| > 0$, and the term Δu_{min}

is a preset small constant used to insure that a nonzero step will be obtained even if all control values at the nth period are zero. Also,

$$\left\| \frac{\partial H_n}{\partial U_n} \right\|_2 = \left[\sum_{i=1}^{I_j} \sum_{j=1}^{J} \left(\frac{\partial H_n}{\partial u_{ij,n}} \right)^2 \right]^{1/2}$$

which is called the Euclidean norm.

7) With ε_n calculate

$$U_n^{(m)} = U_n^{(m-1)} + \varepsilon_n \frac{\partial H_n^{(m-1)}}{\partial U_n} , \qquad n = 1,\ldots,N$$

8) For n = 1,...,N check

$$U_{min,n} \leq U_n^{(m)} \leq U_{max,n}$$

If the inequalities are satisfied for each component of $U_n^{(m)}$, return to 2).

If for any component, the inequalities are not satisfied, reduce ε_n by a factor of 2; and return to 7).

This last step requires some additional programming logic. In particular, if no ε_n of "reasonable" size can be found such that the inequalities are satisfied, then one of two things must be done. Either the component(s) of U_n which are violating the constraints can be set to the value of the nearest bound, or a static optimizer can be called to optimize H_n with respect to U_n. Experiments with EPOC/TIMM have shown that the latter strategy is generally more effective, particularly for nonconvex problems.

In implementing the preceding algorithm it is necessary to compute partial derivatives of the Hamiltonian with respect to U_n and X_{n-1}; and it is at this point that the use of PROSE becomes particularly advantageous, since it provides these partials as a by-product of the evaluation of the objective function. This occurs automatically, and without the need to store derivative formula code.* This is important in two respects. First, for complicated models the partial derivative formulas are difficult to derive; and second, in general, storage requirements become a critical factor in solving problems of the size typically encountered in forest management.

*For more discussion of this process and its implications see [19].

Discussion of the Solution Method

It was pointed out earlier that the maximum principle provides only necessary conditions for an optimum, and that these conditions, are similar to usual first order conditions from calculus. Thus, one suspects, correctly that sufficient conditions for a (local) maximum would be that the matrix of second partial derivatives of H_n with respect to U_n, i.e., the Hessian, must be negative definite at that optimum. This is not especially easy to check, but if this or some equivalent condition is not imposed it is quite possible for the control vector iterations to converge to a minimum or a saddle point, rather than to a maximum. The condition can be removed if the sequence of objective function values is required to be monotonic. In other words, if the goal is to maximize P, then at the end of each iteration it is checked whether

$$P^{(m)} \geq P^{(m-1)} . \qquad (18)$$

Clearly, if $\varepsilon_n = 0$ for all $n = 1,\ldots,N$

this will be satisfied as an equality. If a maximum has been reached at iteration (m-1), then (18) can be satisfied only in this manner. On the other hand if $P^{(m-1)}$ is not a maximum, and (18) is not satisfied, there must exist a sequence $\{\varepsilon_n\}$ such that (18) can be satisfied. The goal of the programming logic of EPOC is to modify the ε_n obtained using (17) so that such a sequence is obtained; and (18) is thus satisfied.

It can be seen from the solution summary in Appendix C that EPOC/TIMM typically takes very few iterations to converge to a solution. There are two main reasons for this. The first pertains to the structure of the physical forest model, while the second is a consequence of the particular implementation of the control vector iteration algorithm.

The forest model, it may be recalled, is derived so that the control vector components represent fractions of available area to which a particular management activity is to be applied. Hence, their values must lie between zero and one. In particular then, their magnitude is always order 1; and this leads to very well-conditioned problems, avoiding numerical difficulties which tend to slc an iterative procedure.

The method by which the step lengths, ε_n, are obtained is the other important factor in the rapid convergence exhibited by EPOC/TIMM. An examination of equation (17) will show that all quantities used in the computation of ε_n are known at the time U_n is to be updated. Specifically, no additional model calls are needed to compute ε_n. This is in contrast to the usual means of calculating step length (cf. [18]). Because it can be calculated so cheaply, a different value of ε_n may be used in each period, as the notation suggests. Thus, in each period the U_n are calculated via a step length which applies specifically to that set of U_n. In more conventional control algorithms one step length would be used for all periods, since obtaining ε is generally quite costly in terms of computation time.