TYPICAL OPTIMAL CONTROL PROBLEM

Maximize (objective)

(1) $$P = \sum_{n=1}^{N} P_n(X_{n-1}, U_n) + G(X_N)$$

Subject to (state equations)

(2) $$X_n = F_n(X_{n-1}, U_n), \quad X_0 \text{ given},$$

and (control bounds)

(3) $$U_{n,min} \leq U_n \leq U_{n,max}$$

The DISCRETE MAXIMUM PRINCIPLE

Theorem In order that $\{U_n\}$ be the optimal controls for the above problem, the following conditions must obtain.

i) either $\dfrac{\partial H_n}{\partial U_n} = 0$, or $\{U_n\}$ maximizes H_n

ii) $\exists \ \lambda_n \ni \ \lambda_N = \dfrac{\partial G}{\partial X_N}$, $\quad \lambda_n = \dfrac{\partial H_{n+1}}{\partial X_n}$

iii) $X_n = \dfrac{\partial H_n}{\partial \lambda_n}$

where

$$H_n = P_n(X_{n-1}, U_n) + \lambda_n F_n(X_{n-1}, U_n)$$

Proof of Discrete Maximum Principle

— Basic idea is to consider problem as static optimization, and apply Kuhn-Tucker conditions.

1. Treat state eqns. as equality constraints,

$$X_n - F_n(X_{n-1}, U_n) = 0,$$

and form Lagrangian:

$$L(X,U;\lambda) = \sum_{n=1}^{N} \{ P_n(X_{n-1},U_n) - \lambda_n [X_n - F_n(X_{n-1},U_n)] \} + G(X_N)$$

2. Set to zero all partials of L wrt X_{n-1}, U_n, λ_n:

$$\frac{\partial L}{\partial X_{n-1}} = \frac{\partial P_n}{\partial X_{n-1}} + \lambda_n \frac{\partial F_n}{\partial X_{n-1}} - \lambda_{n-1} = 0 ; \quad \frac{\partial L}{\partial X_N} = -\lambda_N + \frac{\partial G}{\partial X_N} = 0 ;$$

$$\frac{\partial L}{\partial U_n} = \frac{\partial P_n}{\partial U_n} + \lambda_n \frac{\partial F_n}{\partial U_n} = 0 ; \quad \frac{\partial L}{\partial \lambda_n} = X_n - F_n = 0 ;$$

and use definition of H_n to get

$$\boxed{\lambda_{n-1} = \frac{\partial H_n}{\partial X_{n-1}} , \quad \lambda_N = \frac{\partial G}{\partial X_N} ; \quad \frac{\partial H_n}{\partial U_n} = 0}$$

and $F_n = X_n \Rightarrow \boxed{X_n = \frac{\partial H_n}{\partial \lambda_n}}$

3. Consider a general inequality constraint,

$$g_n(X_{n-1}, U_n) \geqslant 0.$$

Then $\exists$ $z_n \ni$

$$g_n(X_{n-1}, U_n) - z_n^2 = 0$$

$$\therefore \quad z_n - \sqrt{g_n(X_{n-1}, U_n)} = 0$$

Thus, inequalities may be converted to equalities, which can be adjoined to the original Lagrangian using multipliers μ_n. The slack variables, z_n, constitute additional states.

CONTROL VECTOR ITERATION

Algorithm:

1. Guess values for U_n, $n = 1, \ldots, N$.

2. Evaluate $X_n = F_n(X_{n-1}, U_n)$ & $P_n(X_{n-1}, U_n)$

3. Calculate $\lambda_N = \dfrac{\partial G}{\partial X_N}$.

   Then,

   $$H_N = P_N + \lambda_N F_N$$

   $$\Rightarrow \lambda_{N-1} = \frac{\partial H_N}{\partial X_{N-1}}, \quad \frac{\partial H_N}{\partial U_N}$$

4. Successively calculate H_n & $\lambda_{n-1} = \partial H_n / \partial X_{n-1}$

5. IF $\dfrac{\partial H_n}{\partial U_n} \neq 0$, THEN calculate

   $$U_n^{(m)} = U_n^{(m-1)} + \epsilon_n \frac{\partial H_n^{(m-1)}}{\partial U_n}$$

   IF $\dfrac{\partial H_n}{\partial U_n} = 0 \quad \forall\, n = 1, \ldots, N$, STØP.

6. IF $U_{n,min} \leq U_n^{(m)} \leq U_{n,max}$ THEN GØTØ 2.

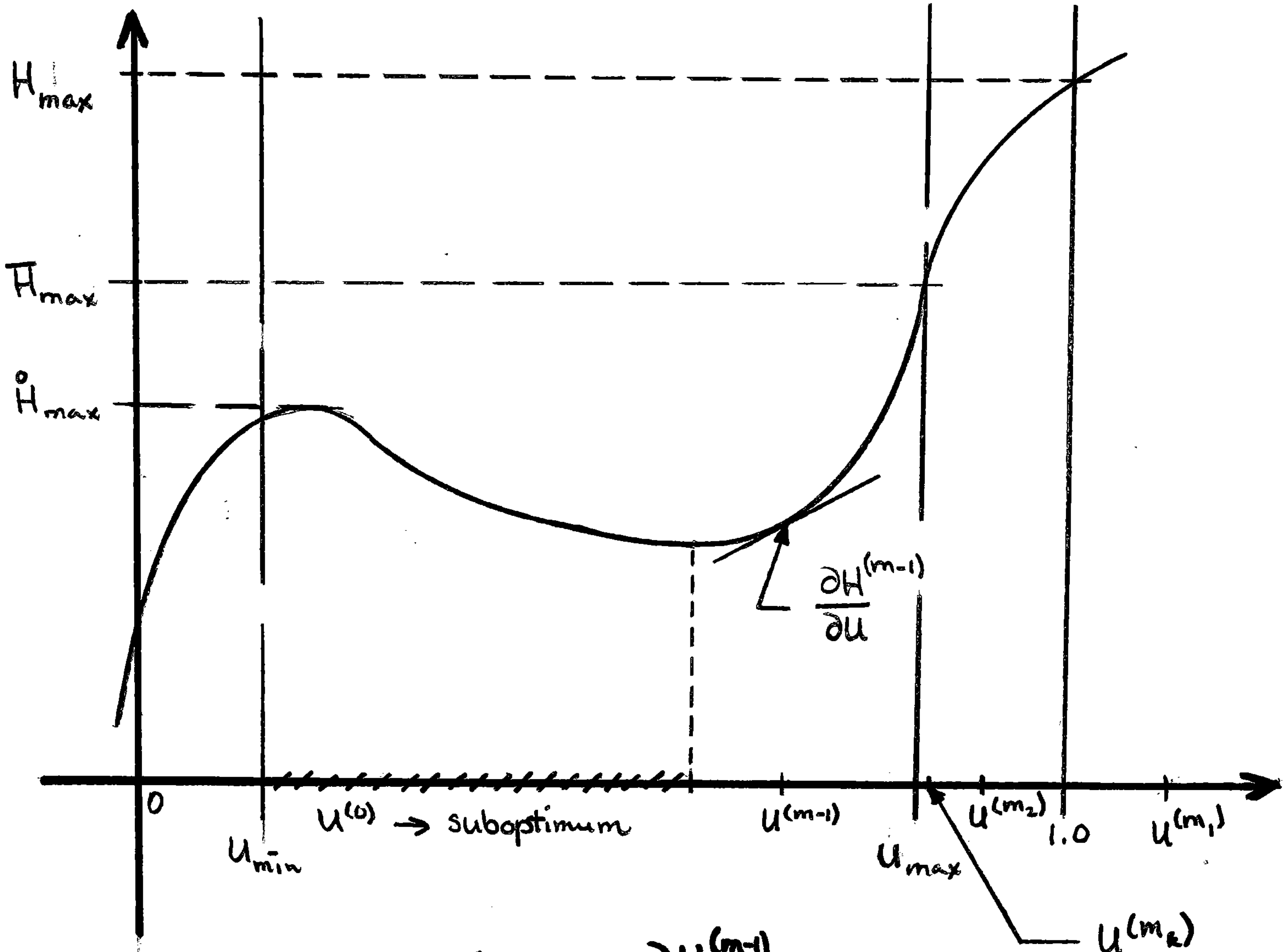
   ELSE $\epsilon_n = \epsilon_n / 2$.

   IF $\epsilon_n > \epsilon_{min}$ THEN GØ TØ 5.

   ELSE $U_n^{(m)} = \begin{cases} U_{n,min} \\ U_{n,max} \end{cases}$

   or call optimizer.

HAMILTONIAN vs. CONTROL



$$U^{(m)} = U^{(m-1)} + \epsilon \frac{\partial H^{(m-1)}}{\partial U}$$

THE MODEL EQUATIONS

Objective Function:

$$P = \sum_{n=1}^{N} \frac{R_n - C_n}{(1+r_e)^n}$$

Transfer Functions:

$$x_{1,n} = x_{1,n-1}(1-u_{1,n}) + \sum_{k=2}^{I} x_{k,n-1}\, u_{k,n}$$

$$x_{2,n} = x_{1,n-1}\, u_{1,n}$$

$$x_{i,n} = x_{i-1,n-1}(1-u_{i-1,n}) \qquad i = 3, \ldots, I.$$

Revenue and Cost Functions:

$$R_n = p V_{H,n}$$

where

$$p = 100 - 0.00005 V_{H,n} / s$$

$$V_{H,n} = \sum_{i=2}^{I} x_{i,n-1} u_{i,n-1} V_{i,n}$$

$$C_n = C_{F,n} + C_{R,n} + C_{H,n}$$

where

$$C_{F,n} = 0.5 A_T s$$

$$C_{R,n} = 30 A_{R,n} = 30 x_{1,n-1} u_{1,n}$$

Harvest (logging) Costs:

$$C_{H,n} = \sum_{i=2}^{I} x_{i,n-1} \, u_{i,n} \, v_{i,n} \, \bar{c}_{Hi,n}$$

where

$v_{i,n}$ ~ volume per acre of age class i, determined from a cubic correlation

$\bar{c}_{Hi,n}$ ~ cost per unit volume of age class i as a function of volume per acre.

Discount Rate:

$$r_e = \left[\frac{sr(1+r)^{ns}}{(1+r)^s - 1} \right]^{1/n} - 1$$

COMPARISON OF MODEL SETUP (Ref. [3] VS. EPOC/TIMM) FOR 100 YEARS

EPOC/TIMM	Ref. [3]
1. Harvest sequence determined as part of solution	1. Oldest trees harvested first
2. Trees younger than 30 yrs. not available for harvest	2. All trees available for harvest
3. 10-year planning periods	3. 1-year planning periods
4. 100% regeneration	4. 100% regeneration