A NONLINEAR OPTIMAL CONTROL MODEL FOR TIMBERLAND MANAGEMENT

by

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ABSTRACT

A discrete optimal control approach is applied to a general timberland management model. A solution technique employing the discrete maximum principle and control vector iteration is presented, and aspects of its implementation are considered. Results of numerically solving a specific model, using a PROSE language code, are given; and possible areas for future development are discussed.

1. INTRODUCTION

A fundamental problem in the management of scarce resources is how to best control the use and renewal of these resources in such a way as to gain an optimal economic return on investments while assuring that adequate supplies of the particular resource will be available at all times in the near, and not so near, future. Particularly, those who manage timberland are confronted with this problem. The purpose of this paper is to present a systematic means of solving such a problem via a discrete optimal control approach. The model given herein deals specifically with timber resources, although it should be clear that only minor modifications would be needed in treating a variety of resource problems.

In the past, linear programming techniques have most often been used in forest management problems. However, these have generally proved inadequate. The forest management problem is inherently nonlinear, thus requiring piecewise linearization if LP models are to be used. But at the same time, forest models typically consist of a large number of variables (often $0(10^4)$); hence, piecewise linearization results in a need to solve extremely large linear programs - large enough to greatly tax the storage capacity of the most modern computers. One approach to this problem is the one usually taken by the U.S. Forestry Service. Namely, economic considerations are ignored, and biological constraints, such as the requirement of even-flow harvest from year to year, are formulated to be linear. A linear program (e.g., TIMBER RAM [1]) can then be used to determine the area to be harvested from each age class (all trees of a specified age) and type site (a land parcel of a certain land classification,* foforested by a particular species, under a

*Land may be classified as to altitude, slope, fertility, etc.

specified cultural treatment*) during each period of the planning horizon. Other linear and quadratic programming approaches are discussed in [2]. These typically incorporate economic factors by employing simplifying assumptions such as perfect competition for raw timber.

Walker [3] presents a nonlinear model which permits the use of any desired demand curve. However, it has been found to be difficult to include additional constraints, particularly those of a biological nature, without significantly altering the rather ad hoc solution scheme. Optimal control models have also been considered previously. Typical examples may be found in the works of Schreuder [4], Näslund [5], and Amidon and Akins [6]. In [4] the problem of optimal thinning and rotation of even-aged forests is approached via dynamic programming. The maximum principle of Pontryagin is used in [5] to provide some analytic insight into such problems as optimal (economically) rotation periods (length of time from one clear-cut harvest to the next succeeding one on a given land parcel) and thinning. In [6], dynamic programming was used to predict optimal growing stock inventories. While these models admit nonlinear formulations, they are still rather limited in the range of problems which they can readily address. In particular, none of the preceding nonlinear models, except that of reference [3], lends itself to use in digital computer solution of problems comparable in size to those of the LP models.

The remainder of this work will be devoted to the formulation of an optimal control model which can be easily modified to handle a wide range of economic/biological problems. Moreover, the method of solution employed is such that, in spite of model nonlinearities, a decomposition can be

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*Cultural treatment includes among other things temporal spacing and intensity of thinning, tree age at clear-cut harvest, and frequency of regeneration (reforestation of bare land).

effected to ensure that problems comparable in size to the LP models may be solved on present computers, if necessary using external storage devices.

2. OPTIMAL CONTROL MODEL

We shall anticipate from the start that a numerical solution will be required, and immediately formulate a discrete model. The usual discrete optimal control initial value problem takes the following general form:

Maximize

$$P = \sum_{n=1}^{N} P_n(X_{n-1}, U_n) + G(X_N) \tag{1}$$

Subject to

$$X_n = F_n(X_{n-1}, U_n), \text{ with } X_0 \text{ given}; \tag{2}$$

and

$$U_{n,min} \le U_n \le U_{n,max}. \tag{3}$$

Equations (2) and the inequalities (3) are vector expressions with equality, or inequality, holding between corresponding components.

In terms of problems in forestry, equation (1) is the management objective; in the sequel this will be net present value over a planning horizon consisting of N planning periods. Equation (2), the state equation, represents the forest inventory at the end of period n. For a forest consisting of I_j age classes on each of J type sites X_n can be expressed in terms of components as

$$X_n = (x_{1,1,n}, x_{2,1,n}, \ldots, x_{I_1,1,n}; \ldots; x_{1,J,n}, \ldots, x_{I_J,J,n}). \tag{4}$$

Similar representations obtain for the vector of transfer functions F_n, and for the controls and control bounds, U_n, $U_{n,min}$, and $U_{n,max}$, respectively. In particular it should be noticed that these are all in one-to-one correspondence. Thus, since each component of X_n represents a specific area of land and each component of U_n is a management activity applied to the corresponding land parcel, it follows that only one management activity may be applied to any single land segment in any one period. Hence, for example, regeneration and harvesting cannot occur on the same parcel in the same planning period.

Each component of X_{n-1} represents an area of land of type site j, forested by timber in age class i at the end of period n-1 (beginning of period n). The corresponding control component represents the fraction of the area $x_{i,j,n-1}$ to which the management activity represented by the control $u_{i,j,n}$ is applied. Thus each component of U_n must satisfy

$$0 \leq u_{i,j,n} \leq 1. \tag{5}$$

The explicit forms of the state equations can now be derived using the foregoing, along with the basic assumption that the span, in years, of an age class is equal to the length of one economic planning period.

The first age class consists of vacant land awaiting regeneration. Thus, the state component for this age class is obtained via the following considerations. At the beginning of period n there are $x_{1,j,n-1}$ units of area of bare land. on type site j. During period n the regeneration control, $u_{1,j,n}$ is applied to this area; so at the end of period n $x_{1,j,n-1}(1-u_{1,j,n})$ units of originally vacant type site j land will still be vacant. Also, during period n, harvesting will presumably have occurred, resulting in more vacant land. The amount of this contribution must be $\sum_{k=2}^{I_j} x_{k,j,n-1}u_{k,j,n}$. Thus, at the end of period n the vacant land associated with the jth type site must be

$$x_{1,j,n} = x_{1,j,n-1}(1 - u_{1,j,n}) + \sum_{k=2}^{I_j} x_{k,j,n-1}u_{k,j,n} \, . \tag{6}$$

From the above it can be inferred that $x_{1,j,n-1}u_{1,j,n}$ units of area are regenerated during period n. But at the end of this period these trees must have advanced one age class. Thus, the second transfer function has the form

$$x_{2,j,n} = x_{1,j,n-1}\, u_{1,j,n} \,. \tag{7}$$

For all other state components the transfer function has the following form:

$$x_{i,j,n} = x_{i-1,j,n-1}(1 - u_{i-1,j,n}) \qquad i = 3,\ldots,I_j \,. \tag{8}$$

This can be deduced by arguments similar to the above, employing the basic assumption, stated earlier. It should be pointed out that the state equations as derived, conserve the area of the forest to which they are applied. Surprisingly, this has seldom been the case in earlier forest models, and a great deal of computational effort is often expended on iterative procedures in order to attain area conservation. A typical example of this is discussed in [2].

Before treating an objective function for this model, we will consider a biological constraint which arises quite often in practice, and which can be handled with just a slight modification to the state equations already obtained. This constraint leads to the problem known in forestry literature as conversion to an even age class forest, which was first solved by Nautiyal and Pearse [7], using linear programming. Starting with a forest with an uneven age class/area distribution, it is desired to convert to an even area distribution, while possibly achieving some economic objective. For a forest to have an even age class distribution, say for type site j, each age class must contain A_{jT}/I_j units of area, where A_{jT} is the total area of type site j under management:

$$A_{jT} = \sum_{i=1}^{I_j} x_{i,j,0} \,. \tag{9}$$

Let

$$b_j = \frac{A_{jT}}{I_j} \,;$$

and suppose $M \leq N$. Then it is required that $x_{i,j,M} = b_j$, $i = 1,\ldots,I_j$. Thus, for $n \geq M$ equations (6), (7), and (8) become

$$2x_{1,j,n} = x_{1,j,n-1}(1 - u_{1,j,n}) + \sum_{k=2}^{I_j} x_{k,j,n-1}\, u_{k,j,n} + b_j$$

$$2x_{2,j,n} = x_{1,j,n-1}\, u_{1,j,n} + b_j$$

$$2x_{i,j,n} = x_{i-1,j,n-1}(1 - u_{i-1,j,n}) + b_j \qquad i = 3,\ldots, I_j$$

or

$$x_{1,j,n} = \frac{1}{2}\left[x_{1,j,n-1}(1 - u_{1,j,n-1}) + \sum_{k=2}^{I_j} x_{k,j,n-1}\, u_{k,j,n} + bj\right] \qquad (10)$$

$$x_{2,j,n} = \frac{1}{2}\, [x_{1,j,n-1}\, u_{1,i,n} + b_j] \qquad (11)$$

$$x_{i,j,n} = \frac{1}{2}\, [x_{i-1,j,n-1}\, (1 - u_{i-1,j,n}) + b_j] \qquad (12)$$

With the natural condition that $u_{I_j,j,n} = 1$* it is easy to show that equations (10), (11) and (12) still conserve the forest area.

It is important to note, however, that in the above form the problem is actually a boundary value problem; and it can be expected (and this can be reasoned on physical grounds also) that there are values of N for which no controls satisfying (5) will be consistent with equations (10), (11), and (12).

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*This condition implies that all trees older than some specified age are automatically clear-cut. There are both economic (Walker [3]) and programming implementation related reasons for using this assumption. Regarding the latter, if the condition were not employed, it would be possible for the forest age class array to expand indefinitely.

In theory conditions for controllability can be derived (cf. Bryson and Ho [8]); but for models as complicated as the present one will be seen to be, these conditions are quite difficult to check. In such cases one must generally be guided somewhat by experience and intuition in selecting values of N and M. Of course, N is not completely free in any case since it is associated with the length of the planning horizon; it is this fact that leads to physical arguments concerning controllability for the above case. Namely, if N implies a planning horizon too short for all areas to be clear-cut harvested at least once, then generally (10), (11), and (12) cannot hold.

We will now discuss some specific examples of equation (1). Earlier we mentioned that the model given herein would have as its purpose the maximization of net present value. Use of equations (6), (7) and (8) with the equations to be presented below leads to a purely economic objective. On the other hand, using (10), (11), and (12) while maximizing net present value implies mixed economic/biological goals, exactly the problem treated in [7]. Finally, biological goals may be considered alone by using (6), (7), and (8) along with, for example, only the last term of (1). In particular, we might have

$$\text{minimize } P = G(X_N) = \sum_{i=1}^{I_j} \sum_{j=1}^{J} (x_{i,j,N} - b_j)^2 . \tag{13}$$

This would minimize the deviation from even flow after N periods with no regard for economic consequences.

Many other forms of (1) may be used, but for the remainder of this discussion the following objective will be considered:

Maximize

$$P = \sum_{n=1}^{N} \frac{R_n - C_n}{(1 + r_e)^n} , \tag{14}$$

where R_n and C_n are respectively revenue and cost during period n; and r_e is an equivalent discount rate accounting for the fact that the planning periods may be greater in length than one year. This can be calculated in a number of ways, depending upon the economic assumptions which one makes.

The revenue in period n must depend upon the volume of timber sold and on the price paid per unit of timber. Thus,

$$R_n = \sum_{i=2}^{I_j} \sum_{j=1}^{J} p_{i,j,n} V_{i,j,n}, \tag{15}$$

where $p_{i,j,n}$ is the price (in dollars) per unit volume of timber from the ith age class on type site j in period n; and $V_{i,j,n}$ is the total volume of such timber harvested. (It is assumed that all timber harvested in period n is sold in period n.) $V_{i,j,n}$ is calculated from

$$V_{i,j,n} = x_{i-1,j,n-1} u_{i-1,j,n} v_{i,j}, \tag{16}$$

with $v_{i,j}$ being the volume per unit area of age class i timber on type site j. The general shape of the volume per unit area function is shown in Figure 1. This can be justified from physical arguments based on the known growth characteristics of lumber producing tree species and is corroborated by physical data given in [9]. Such a curve shape can be obtained using a function of the form

$$v_{i,j} = v_{max,j} e^{-\alpha_j/g(u,t)}, \tag{17}$$

where g(u,t) is a homogeneous nonnegative function of the management activities (e.g., fertilization) applied to type site j up to t = (n-1)s; and $v_{max,j}$ is the maximum height which, for biostructural reasons, can ever be attained by trees of the species used in foresting type site j. The constant

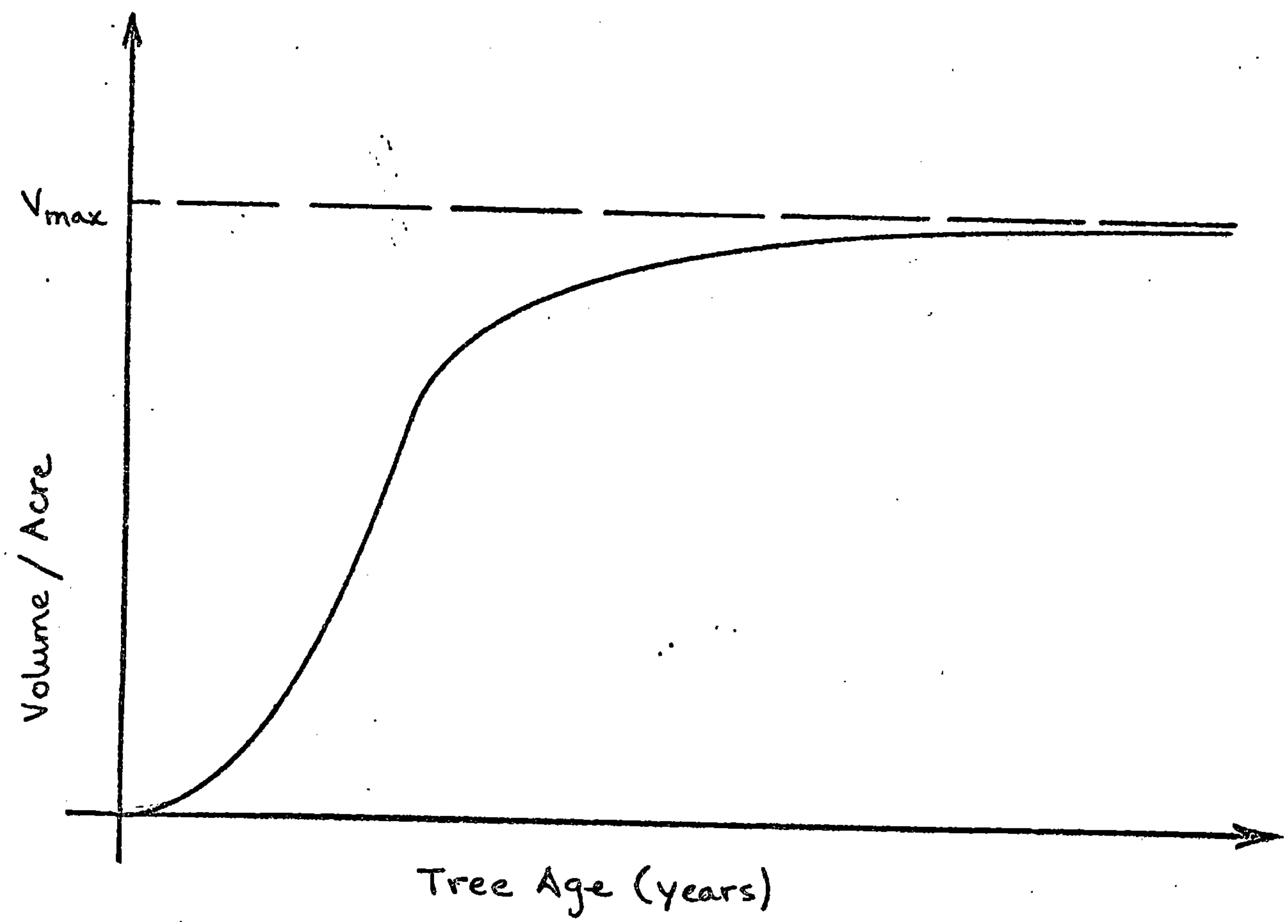

Figure 1. Timber Volume/Acre vs. Tree Age

α_j can be obtained from correlation of data such as those given in [9], and s is the length (in years) of a single planning period.

The price $p_{i,j,n}$ used in equation (15) is usually either assumed constant, as in the linear programs of [2], or is assumed to vary linearly with volume as in the quadratic programs of [2],or the model in [3]. In the present formulation, no particular assumption need be made as to the functional form of $p_{i,j,n}$. It may vary with time as the subscript n implies (and as prices surely must); and it may be different for each different age class and type site. However, in general, $p_{i,j,n}$ would be expected to be some function of $V_{i,j,n}$. Moreover, because of the nature of the solution technique, to be discussed below, $p_{i,j,n}$ should be differentiable with respect to $V_{i,j,n}$.

The cost term in equation (14) is less easily specified. It can be seen to consist of at least three principal components: fixed costs, regeneration costs, and harvest costs. If the forest owner also owns milling operations, which is usually the case for large private ownerships in the timber industry, then a fourth major cost, production costs (which also have fixed and variable components) must be added to the above three. Thus, in general,

$$C_n = C_{F,n} + C_{R,n} + C_{H,n} + C_{P,n}. \qquad (18)$$

Typical forms for each term on the right hand side of (18) are shown graphically in Figure 2. Generally each term may itself depend on a variety of costs, some or all of which may vary with time. Associated with fixed costs, $C_{F,n}$, one would expect such items as salaries, land costs (lease or rent and property taxes), cost of depreciation and maintenance of equipment, etc. The level of fixed costs is strongly influenced by the area of land managed, so

$$C_{F,n} = C_{F,n}(A_T) \ . \qquad (19)$$

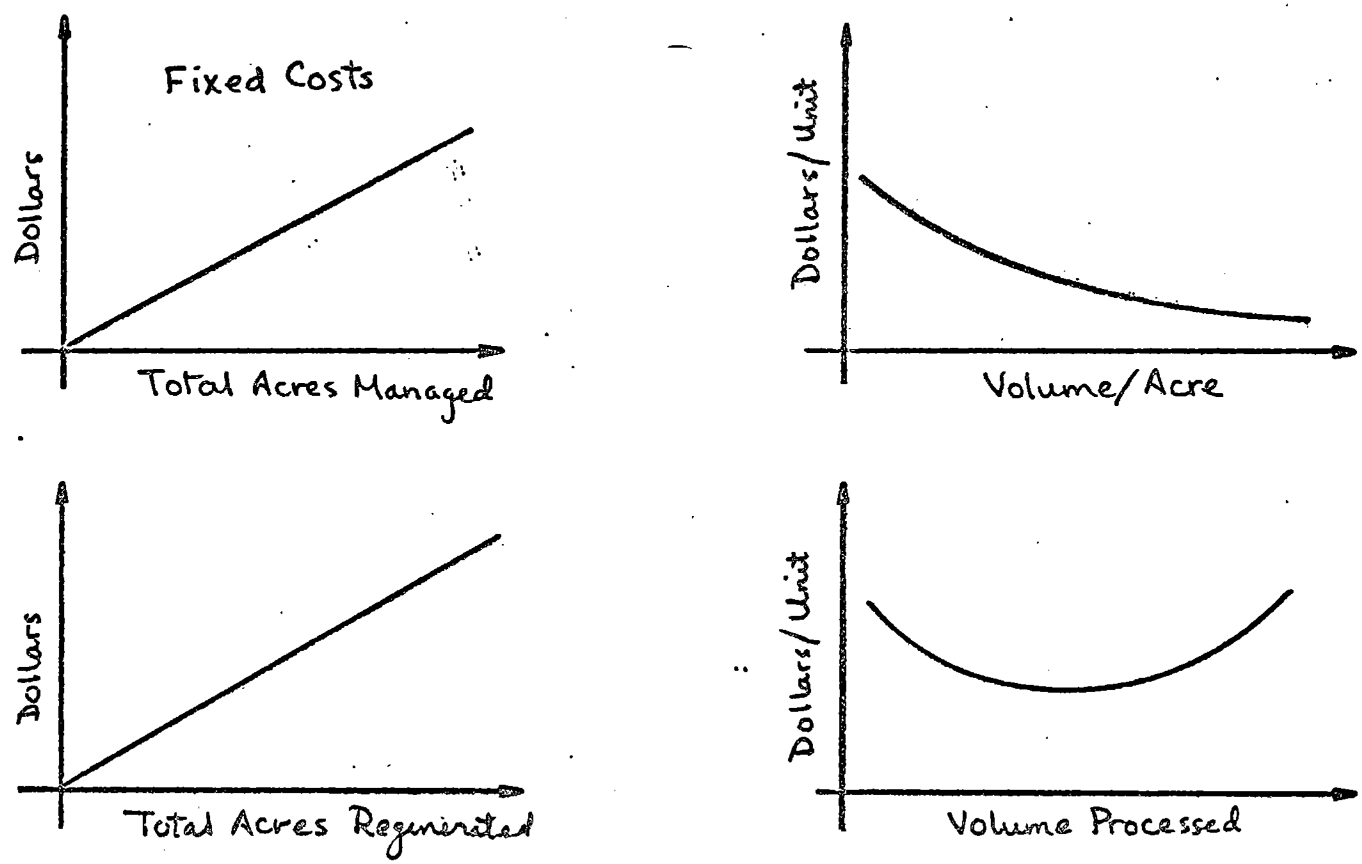

Figure 2. Typical Cost Functions

Regeneration costs are usually assumed to be directly proportional to the area of land regenerated, $A_{R,n}$, depending mainly on the cost of seeding. Thus,

$$C_{R,n} = C_{R,n}(A_{R,n}) \tag{20}$$

with

$$A_{R,n} = \sum_{j=1}^{J} x_{1,j,n-1}u_{1,j,n}. \tag{21}$$

Harvest costs, $C_{H,n}$, clearly depend on the volume of timber harvested, due to increases in labor and transportation costs with increases in harvest volumes. However, costs are usually first computed in terms of volume per area, so that as the volume per area increases, the costs are amortized over larger volumes, decreasing the cost per unit of timber harvested. This accounts for the shape of the corresponding curve in Figure 2. Of course, the total cost of harvest must increase with volume. Analogous to (15),

$$C_{H,n} = \sum_{i=2}^{I_j} \sum_{j=1}^{J} c_{H,i,j,n}V_{i,j,n}$$

where $c_{H,i,j,n}$ is a function of the volume, $V_{i,j,n}$. (22)

Finally, production costs $C_{P,n}$ are primarily those costs associated with the operation of sawmills, pulp processing plants, etc. The operation costs will usually rise significantly when the plants are operating very far from their design capacity, as is depicted in Figure 2. Thus, $C_{P,n}$ must be a function of harvest volume. This might be further segmented, with costs at particular plants being influenced only by volume harvested from specific age classes and type sites, or into fixed and variable costs. For example, operating costs in a lumber mill should not be influenced by the harvest volume of pulpwood, unless this itself, in some way, dictates the amount of timber which could be harvested for lumber production. An implicit expression for production costs is then

$$C_{P,n} = C_{P,n}(V_{H,n}), \tag{23}$$

where $V_{H,n}$ is the vector of harvest volumes having the $V_{i,j,n}$ as components. This completes a rather general specification of the timberland management model. Before proceeding to a discussion of the method used to solve this model, a few remarks, pertaining to the various restrictions and assumptions noted earlier, are of interest. In all cases, these restrictions have been imposed primarily to simplify the model equations. Requiring a one-to-one correspondence between controls and land parcels leads to the specification of management activities on each age class/type site parcel in terms of a scalar for each period. Lifting of this restriction would lead to a vector representation. By assuming that the span of each age class is equal to the length of an economic planning period, we have avoided dealing with transferring fractions of age classes between planning periods. From the development leading to equation (6), it can be inferred that land vacated by any level of harvesting (thinning is considered to be harvesting at a level less than clear-cutting) is returned immediately to the regeneration land pool. A more complicated expression similar to (6), which does not contain this implication, can be derived. But it should be noted that land in the regeneration pool need not be immediately regenerated, so it is doubtful that an expression more general than (6) is really necessary. As mentioned earlier, equation (6) also implies that for trees older than some specified maximum age, clear-cutting always occurs. An alternative is to consider a final class of trees which contains all ages above a specified age, and for which it is assumed that no further growth can occur. A slight modification to equation (6) and the I_jth equation of (8) can be made for such a treatment. Finally, the assumption that all timber harvested in a particular planning period is sold in that period was made to avoid keeping track of inventories. Clearly, a more complicated objective model need not incorporate such an assumption.

3. SOLUTION TECHNIQUE

There are at least three possible approaches to the solution of the model formulated in the preceding section: 1) use of linear programming with piecewise linearization and Dantzig-Wolfe decomposition, 2) dynamic programming, and 3) the discrete maximum principle.

The first of these, piecewise linear programming, has the obvious disadvantage that in no case is the original nonlinear problem actually solved. Furthermore, both storage difficulties and excessive run times can be encountered for large problems. Use of Dantzig-Wolfe decomposition on a problem from forestry is discussed in Lasdon [10]; and it appears from the data presented that such decomposition schemes may require a very large number of iterations. While dynamic programming has proved valuable in analytical work, and in the numerical treatment of small problems, it eventually leads to impossible storage requirements, a manifestation of the "curse of dimensionality," as well as long run times. The discrete maximum principle provides a means of surmounting the storage problem by decomposing the original problem into separate subproblems, in a manner somewhat analogous to the Dantzig-Wolfe decomposition mentioned above for linear programming. Moreover, just as with dynamic programming, the original nonlinear problem is treated using the discrete maximum principle. The major disadvantages are long run times and possible convergence to a suboptimum. It is clear that none of the preceding methods is very satisfactory; but many of the difficulties lie as much in the problem complexity as with inherent shortcomings of the solution procedures. On balance, it is felt that the discrete maximum principle is the best of the three methods. In this section the means to apply it to the model developed earlier will be discussed.

The discrete maximum principle provides a set of necessary conditions for attaining an optimum sequence of controls $\{U_n\}$ for the basic discrete optimal control problem typified by equations (1) and (2), and the inequalitites of (3). It may be obtained by formally discretizing the continuous version of the maximum principle of Pontryagin [11]. Proofs of the discrete maximum principle are usually based on the application of the Kuhn-Tucker conditions to the Lagrangian formed by treating the state equations (2) as equality constraints, and introducing Lagrange multipliers, λ_n (variously termed co-states, covariables, and adjoint variables) to adjoin these to the original objective, equation (1). Such an approach is used in [12] where the main theorem appears in virtually the same form as will be given below. It does not seem necessary to repeat the proof here.

Theorem (Discrete Maximum Principle)

In order for the sequence of vectors $\{U_n\}$ to be the optimal controls for the problem represented by equations (1) and (2) and inequality (3), the following conditions must obtain:

i) Either $\partial H_n/\partial U_n = 0$,
   or $\{U_n\}$ is such that H_n is optimal
   for all $n = 1,\ldots,N$.

ii) There exists a sequence of vectors $\{\lambda_n\}$ such that

$$\lambda_N = \frac{\partial G}{\partial X_N}\ , \text{ and } \lambda_n = \frac{\partial H_{n+1}}{\partial X_n} \qquad n = N-1,\ldots,1.$$

iii) The sequence of state vectors $\{X_n\}$ satisfies

$$X_n = \frac{\partial H_n}{\partial \lambda_n} \qquad n = 1, \ldots, N.$$

In the above conditions H_n is the Hamiltonian for period n, defined as

$$H_n(X_{n-1}, U_n; \lambda_n) = P_n(X_{n-1}, U_n) + \lambda_n F_n(X_{n-1}, U_n) \ . \tag{24}$$

Before considering the conditions of the theorem in more detail, a remark concerning the notation is in order. Since U_n, X_n, and λ_n are all vectors, the corresponding differential operators, e.g., $\frac{\partial}{\partial U_n}$ are actually gradients. Also, the multiplication indicated by $\lambda_n F_n$ in (24) is computed as an inner product. (In reality it is a generalization of the inner product, the evaluation of the 1-form λ_n at the vector F_n, often denoted $\langle \lambda_n, F_n \rangle$.)

The conditions implied by i) indicate that necessary conditions are provided for both interior optima, and optima on the control boundaries. It is the ability to treat the latter which distinguishes the maximum principle (and also dynamic programming) from the calculus variations.* It is the content of condition i) which allows the periodwise decomposition of serial problems, that is, the conditions for an optimum apply to each Hamiltonian, individually. Condition ii) is extremely important, for it is the existence of the Lagrange multipliers which permits the "transmission of information" between stages in the decomposed problem. Finally, it is easy to show using (24) that condition iii) corresponds to ensuring that the state equations hold.

*A lucid and concise comparative treatment of these three methods is given in Intriligator [13].

If one constructs the Hamiltonian, and then attempts to apply conditions i), ii), and iii) via algebraic manipulations, it quickly becomes evident that the only practical implementations of the discrete maximum principle must be numerical. For problems of the form treated in this work, a natural realization is through control vector iteration. The basic procedure is outlined in the following.

Algorithm (Control Vector Iteration)

i) Guess values of U_n, $n = 1,\ldots,N$; and set $m = 1$

ii) Evaluate state equations and objective for all $n = 1,\ldots,N$

If $m > 1$, check convergence criteria.* If satisfied, stop.

Else continue to iii).

iii) Using X_N obtain $\lambda_N = \frac{\partial G}{\partial X_N}$,

iv) Calculate H_N, then λ_{N-1}, then H_{N-1}, etc.

v) a: If $\frac{\partial H_n}{\partial U_n} = 0$ for all $n = 1,\ldots,N$, stop.

b: Else update U_n using the standard gradient stepping procedure from static optimization:

$$U_n^{(m)} = U_n^{(m-1)} + \varepsilon_n \frac{\partial H_n}{\partial U_n}^{(m-1)} \qquad (25)$$

where m is the iteration number; and $\varepsilon > 0$ is the step size, which may be computed in a variety of ways.

*The condition used for the current code was $P^{(m)} - P^{(m-1)} > \delta$.

vi) If $U_{min} \leq U^{(m)} \leq U_{max}$, go to vii)

Else $\varepsilon = \varepsilon/2$

If $\varepsilon > \varepsilon_{min}$ go to v)b.

$$\text{Else } U_n^{(m)} = \begin{cases} U_{n,min} & \text{if} \quad U_n^{(m)} < U_{n,min} \\ U_{n,max} & \text{if} \quad U_n^{(m)} > U_{max,n} \end{cases}$$

or perform optimization of H_n with respect to U_n

vii) Increment m to m + 1, and go to ii).

From equation (25) it is clear that this is basically a steepest descent method. Such methods have the advantage of relatively small storage requirements and fewer computations per iteration than second order methods. On the other hand, without the information contained in the eigenvalue structure of the Hessian, convergence to the desired type of optimum can only be guaranteed for convex objectives, and in any case may be slow. However, in the present case, convergence is aided by two factors. First, the control values all lie on the unit interval. This is particularly advantageous when the model equations are properly scaled. Second, the step length ε_n of equation (25) is computed using a very simple expression involving only information previously generated. Hence, it can be calculated cheaply; and thus a value of ε_n is computed specifically for each individual period n, as the notation suggests.

For problems as large and complicated as the model presented herein, any method requiring derivative evaluation can be justly criticized, since at least until recently derivatives could be obtained only by finite differencing, or by formula code. The latter, whether generated by hand, or by automatic formula manipulation could require significant storage allocation, while the former leads to inaccuracy and poor convergence properties. To avoid these difficulties, the algorithm was coded in the PROSE language* which provides exact partial derivatives automatically without the need to store derivative formulas.**

Another feature of the present implementation is the explicit use made of the decomposition provided by the discrete maximum principle. In particular only computations for one planning period are kept in fast access memory. All results unnecessary to immediate calculations are stored on disc files, and recalled as needed. While this use of external storage is a time consuming process, it is the only feasible way to handle the very large problems encountered in forest management, where in excess of 10^4 control variables may have to be determined.

4. RESULTS AND CONCLUSIONS

The model described in Reference [3] may be reformulated as a discrete optimal control problem. When this is done, a specific case of the general model discussed above is obtained (see ref. [16] for details). In particular, a forest consisting of only one type site (J=1) was considered, utilizing a single linear demand function, p, and only three components in the cost function. Plant production costs were not considered. The model was solved for a 100-year planning horizon, subdivided into 10 periods of equal length, during which the net present value of an appproximately 250,000 acre forest was to be maximized starting from a prescribed initial state. No specific biological constraints were imposed on the solution except that all vacant land available for regeneration at the beginning of each planning period was required to be regenerated during that period. In addition, harvest of trees younger than 30 years of age was not permitted.

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*PROSE is a new very high level, general purpose language described in some detail by Thames [14].

**The method is based on a technique due to Wengert [15].

All results* (summarized by Figures 3 and 4) were in excellent agreement with those produced by the model of [3], the difference in net present values between the two models being only 0.6 percent, for 100 years. But two items merit comment. First, although the net present value was essentially the same for the two models, the harvesting intensities and timing were rather different for specific age classes, but yet such that total volume harvested per period (Figure 3) was essentially the same for both models. This implies that $\{U_n\}$ is not unique, for this particular model. Second, including a production cost function would be expected to more evenly distribute harvest volumes over time; and the inclusion of such a function might also lead to a unique optimal control sequence $\{U_n\}$.

In all, 100 control variables were calculated, corresponding to harvesting intensities on 10 different tree age classes in each of the 10 planning periods. While this is actually a very small problem, by usual forestry standards, it was solved using the algorithm discussed in Section 3 just as a much larger problem would have been solved. Namely, the overall problem was decomposed into 10 subproblems, one for each period within the planning horizon. These were solved one at a time with the individual results not kept in central memory, but rather written on disk files. The solutions for the individual periods were then coordinated using the Lagrange multipliers. The number of planning periods, which may be considered is essentially unlimited, while the number of control variables calculated in each period can probably be as great as approximately 100, the exact number depending on the complexity, and thus storage requirements, of a specific model.

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*For a more detailed discussion and full presentation of results of this problem, as well as those of a related problem, see reference [16].

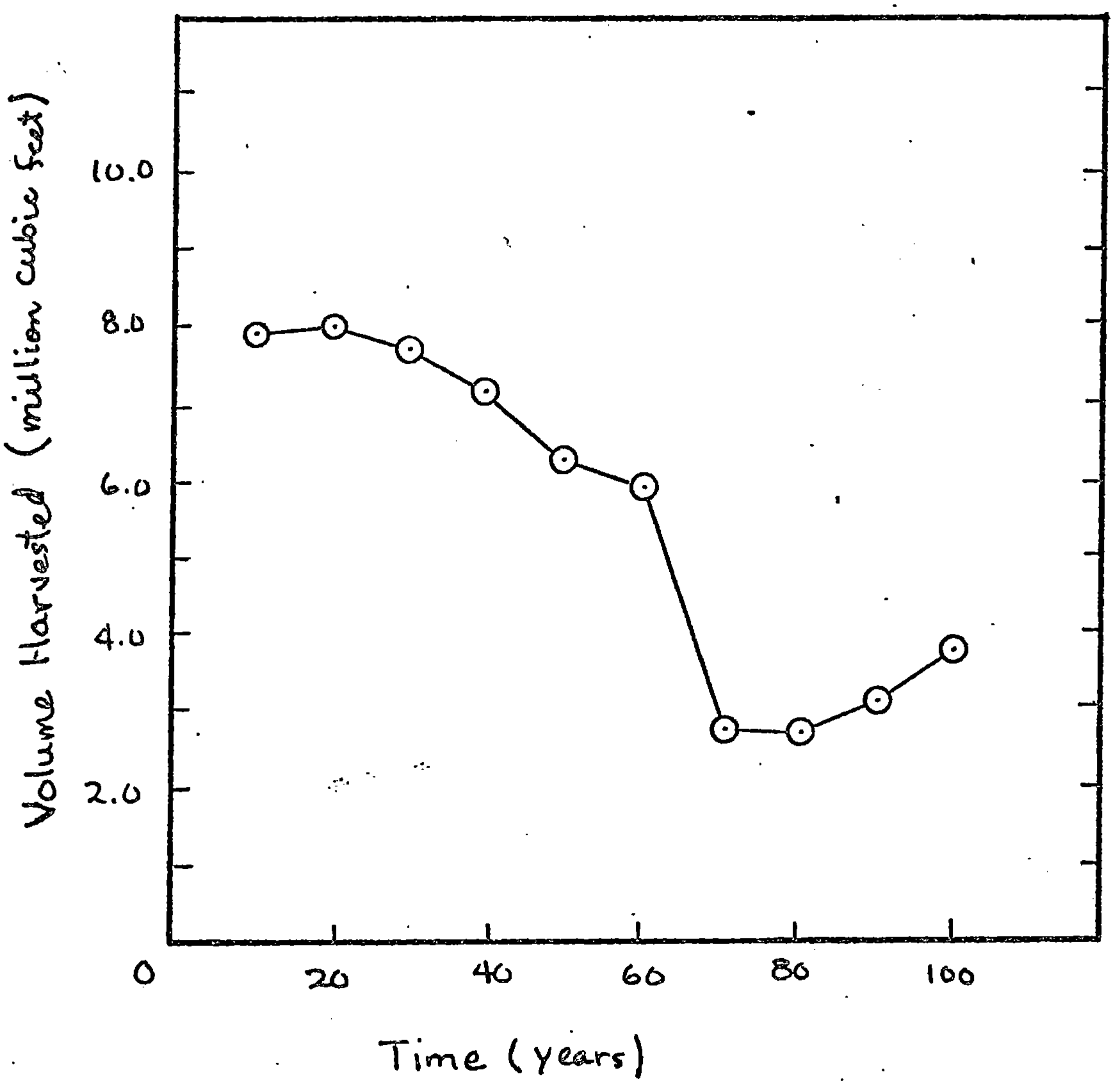

Figure 3. Volume of Timber Harvested vs. Time

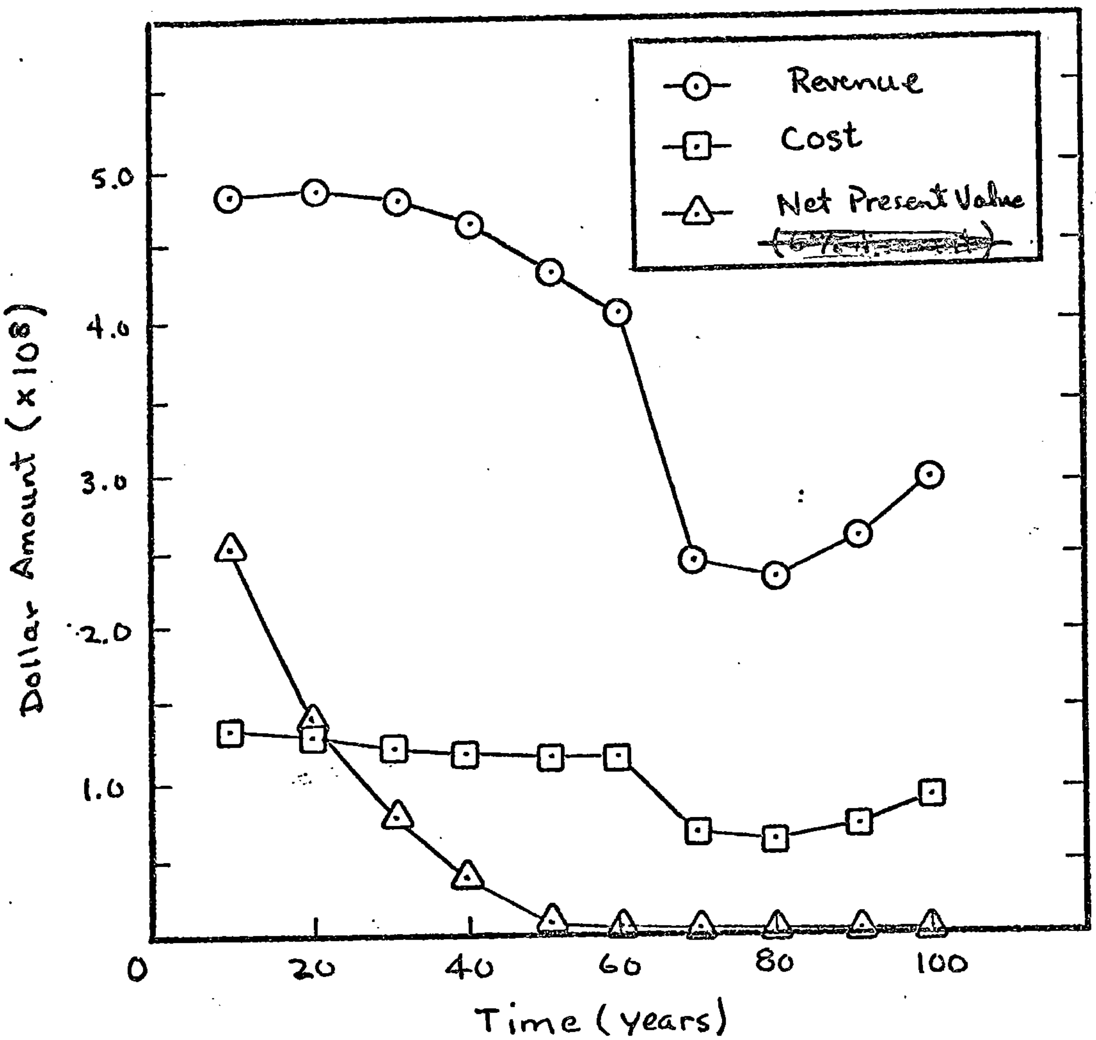

Figure 4. Revenue, Cost, & Net Present Value vs Time for 6 % Discount Rate

We feel that the main contribution of the optimal control approach to forest management, as presented herein, is that it provides a very flexible and workable tool for solving a wide variety of problems in timberland management, and resource management in general. Very general, and hopefully realistic, objectives including both economic and ecological constraints may be considered because the storage requirement difficulty often encountered with large nonlinear problems has been avoided by utilizing the natural decomposition provided by discrete maximum principle/ control vector iteration. In addition, another significant problem usually associated with nonlinear programming, that of obtaining partial derivatives of the objective and constraint functions, is handled easily by employing the PROSE language.

While optimal control solutions to this type of problem are not new, the implementation of such techniques in solving very large problems by computer is new; and much further work merits attention. We have given very little consideration to such questions as existence and uniqueness of solutions. Moreover, the algorithm presented above applies only to initial value problems. Thus, there is much to be done on boundary value problems. For example, one might run a coarse initial value optimization for the entire planning horizon, and then use boundary value techniques to improve solutions in particular periods of interest; questions of controllability and reachability must then be considered. Much more complicated models than the one presented in Section 2 might be treated. Stochastic processes could come into play if various risks in investment in timberland are considered. Markov decision chain approaches to this type problem are exemplified by the works of Hool [17] and Lembersky and Johnson [18]. The model might also be generalized in another sense; namely, so as to also optimize the distribution of timber products. This is attempted, but not with complete success, in the linear model of reference [19].

We thus conclude that the current model provides a starting point, and a direction, for much further study into the computational aspects of the optimal use of renewable natural resources in general, and timberland in particular, via nonlinear optimal control models.

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